

Chapter 1

Exact solutions for pairing interactions

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The exact solution of the BCS pairing Hamiltonian was found by Richardson in 1963. While little attention was paid to this exactly solvable model in the remainder of the 20th century, there was a burst of work at the beginning of this century focusing on its applications in different areas of quantum physics. We review the history of this exact solution and discuss recent developments related to the Richardson-Gaudin class of integrable models, focussing on the role of these various models in nuclear physics.

1. Cooper pairs, BCS and the Richardson exact solution

The first breakthrough towards a microscopic description of the superconducting phenomenon was due to Cooper¹, who in 1956 showed that a single pair of electrons on top of an inert Fermi sea could be bound by an infinitesimal attractive interaction. The search for a many-body wave function describing a fraction of correlated and overlapping pairs mixed with a Fermi sea was a key goal for the rest of that year. Schrieffer came up with a solution at the beginning of 1957 and the BCS team (Bardeen, Cooper and Schrieffer) started an intensive and fruitful collaboration to explain quantitatively many superconducting properties from the associated BCS wave function. This led to the famous BCS paper² which provided a complete microscopic explanation of superconductivity.

The success of the BCS theory quickly spread to other quantum many-body systems, including the atomic nucleus. In the summer of 1957, David Pines visited the Niels Bohr Institute and gave a series of seminars about

the yet unpublished BCS theory. Soon thereafter Bohr, Mottelson and Pines published a paper³ suggesting that the gaps observed in even-even nuclei could be due to superconducting correlations. They noted, however, that these effects should be strongly influenced by the finite size of the nucleus. Since then, and up to the present, number projection and in general symmetry restoration in the BCS and Hartree-Fock-Bogoliubov approximations have been important issues in nuclear structure.

At the beginning of the sixties, while several groups were developing numerical techniques for number-projected BCS calculations^{4,5}, Richardson provided an exact solution for the reduced BCS Hamiltonian^{6,7}. In spite of the importance of his exact solution, this work did not have much impact in nuclear physics with just a few exceptions. Later on, his exact solution was rediscovered in the framework of ultrasmall superconducting grains⁸ where BCS and number-projected BCS were unable to describe appropriately the crossover from superconductivity to a normal metal as a function of the grain size. Since then, there has been a flurry of work extending the Richardson exact solution to families of exactly-solvable models, now called the Richardson-Gaudin (RG) models⁹, and applying these models to different areas of quantum many-body physics including mesoscopic systems, condensed matter, quantum optics, cold atomic gases, quantum dots and nuclear structure¹⁰. In this paper, we review Richardson's solution, its generalization to the exactly-solvable RG models and discuss the applications of these models in nuclear physics.

2. The Richardson solution of the reduced BCS hamiltonian

We will focus on a pairing Hamiltonian with constant strength G acting in a space of doubly-degenerate time-reversed states (k, \bar{k}) ,

$$H_P = \sum_k \epsilon_k c_k^\dagger c_k - G \sum_{k, \bar{k}'} c_k^\dagger c_{\bar{k}}^\dagger c_{\bar{k}'} c_{k'} , \quad (1)$$

where ϵ_k are the single-particle energies for the doubly-degenerate orbits k, \bar{k} .

Cooper considered the addition of a pair of fermions with an attractive pairing interaction on top of an inert Fermi sea (FS) under the influence of this Hamiltonian. He showed that the pair eigenstate is

$$|\Psi_{Cooper}\rangle = \sum_{k > k_F} \frac{1}{2\epsilon_k - E} c_k^\dagger c_{\bar{k}}^\dagger |FS\rangle , \quad (2)$$

where E is the energy eigenvalue. Cooper found that for any attractive value of G , the Fermi sea is unstable against the formation of such bound pairs. Therefore, an approach that takes into account a fraction of these correlated pairs mixed with a Fermi sea should be able to describe the superconducting phenomenon.

The BCS approach followed a somewhat different path to the one suggested by Cooper, defining instead a variational wave function as a coherent state of pairs that are averaged over the whole system,

$$|\Psi_{BCS}\rangle = e^{\Gamma^\dagger} |0\rangle, \quad (3)$$

where $\Gamma^\dagger = \sum_k z_k c_k^\dagger c_{\bar{k}}^\dagger$ is the coherent pair. Though errors due to the non-conservation of particle number in (3) are negligible when the number of pairs is sufficiently large, they can be important in such finite systems as atomic nuclei³. To accommodate these effects, number-projected BCS (PBCS)⁴ considers a condensate of pairs of the form

$$|\Psi_{PBCS}\rangle = (\Gamma^\dagger)^M |0\rangle, \quad (4)$$

where M is the number of pairs and Γ^\dagger has the same form as in BCS.

Richardson⁶ proposed an ansatz for the exact solution of the pairing Hamiltonian (1) that followed closely Cooper's original idea. For a system with $2M + \nu$ particles, with ν of these particles unpaired, his ansatz involves a state of the form

$$|\Psi\rangle = B_1^\dagger B_2^\dagger \cdots B_M^\dagger |\nu\rangle, \quad (5)$$

where the collective pair operators B_α^\dagger have the form found by Cooper for the one-pair problem,

$$B_\alpha^\dagger = \sum_{k=1}^L \frac{1}{2\varepsilon_k - E_\alpha} c_k^\dagger c_{\bar{k}}^\dagger. \quad (6)$$

Here L is the number of single-particle levels and

$$|\nu\rangle \equiv |\nu_1, \nu_2, \dots, \nu_L\rangle \quad (7)$$

is a state of ν unpaired fermions ($\nu = \sum_k \nu_k$, with $\nu_k = 1$ or 0) defined by $c_k c_{\bar{k}} |\nu\rangle = 0$, and $n_k |\nu\rangle = \nu_k |\nu\rangle$.

In the one-pair problem, the quantities E_α that enter (6) are the eigenvalues of the pairing Hamiltonian, *i.e.*, the *pair energies*. Richardson proposed to use the M pair energies E_α in the many-body wave function of

Eqs. (5, 6) as parameters which are chosen to fulfill the eigenvalue equation $H_P |\Psi\rangle = E |\Psi\rangle$. He showed that this is the case if the pair energies satisfy a set of M non-linear coupled equations

$$1 - G \sum_{k=1}^L \frac{1 - \nu_k}{2\varepsilon_k - E_\alpha} - 2G \sum_{\beta(\neq\alpha)=1}^M \frac{1}{E_\beta - E_\alpha} = 0, \quad (8)$$

which are now called the Richardson equations. The second term represents the interaction between particles in a given pair and the third term represents the interaction between pairs. The associated eigenvalues of H are given by

$$\mathcal{E} = \sum_{k=1}^L \varepsilon_k \nu_k + \sum_{\alpha=1}^M E_\alpha, \quad (9)$$

namely as a sum of the pair energies.

Each independent solution of the set of Richardson equations defines a set of M pair energies that completely characterizes a particular eigenstate (5, 6). The complete set of eigenstates of the pairing Hamiltonian can be obtained in this way. The ground state solution is the energetically lowest solution in the $\nu = 0$ or $\nu = 1$ sector, depending on whether the system has an even or an odd number of particles, respectively.

There are a couple of points that should be noted here. First, in contrast to the BCS solution, each Cooper pair B_α^\dagger is distinct. Second, if one of the pair energies E_α is complex, then its complex-conjugate E_α^* is also a solution. From this latter point we see that $|\Psi\rangle$ preserves time-reversal invariance.

On inspection of the Richardson pair (6), we see that a pair energy that is close to a particular $2\varepsilon_k$, *i.e.* close to the energy of an unperturbed pair, is dominated by this particular configuration and thus defines an uncorrelated pair. In contrast, a pair energy that lies sufficiently far away in the complex plane produces a correlated Cooper pair. This is to be contrasted with the single BCS coherent pair, which has amplitude $z_k = v_k/u_k$ and which mixes correlated and uncorrelated pairs over the *whole* system.

3. Generalization to the Richardson-Gaudin class of integrable models

In this section, we discuss how to generalize the standard pairing model, which as we have seen is exactly solvable, to a wider variety of exactly-solvable models, the so-called Richardson-Gaudin models¹¹, all of which are

based on the $SU(2)$ algebra. We first introduce the generators of $SU(2)$, using a basis more familiar to nuclear structure,

$$K_j^0 = \frac{1}{2} \left(\sum_m a_{jm}^\dagger a_{jm} - \Omega_j \right), \quad K_j^+ = \sum_m a_{jm}^\dagger a_{j\bar{m}}, \quad K_j^- = (K_j^+)^\dagger. \quad (10)$$

Here a_{jm}^\dagger creates a fermion in single-particle state jm , $j\bar{m}$ denotes the time reverse of jm , and $\Omega_j = j + \frac{1}{2}$ is the pair degeneracy of orbit j . These operators fulfill the $SU(2)$ algebra $[K_j^+, K_{j'}^-] = 2\delta_{jj'} K_j^0$, $[K_j^0, K_{j'}^\pm] = \pm\delta_{jj'} K_{j'}^\pm$.

We now consider a general set of L Hermitian and number-conserving operators that can be built up from the generators of $SU(2)$ with linear and quadratic terms,

$$R_i = K_i^0 + 2g \sum_{j(\neq i)} \left[\frac{X_{ij}}{2} (K_i^+ K_j^- + K_i^- K_j^+) + Y_{ij} K_i^0 K_j^0 \right]. \quad (11)$$

Following Gaudin¹², we then look for the conditions that the matrices X and Y must satisfy in order that the R operators commute with one another. It turns out that there are essentially two families of solutions, referred to as the rational and hyperbolic families, respectively.

i. The rational family

$$X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j} \quad (12)$$

ii. The hyperbolic family

$$X_{ij} = 2 \frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Y_{ij} = \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \quad (13)$$

Here the set of L parameters η_i are free real numbers.

The traditional pairing model is an example of the rational family. It can be obtained as a linear combination of the integrals of motion, $H_P = \sum_j \varepsilon_j R_j(\varepsilon_j)$, with $\eta_j = \varepsilon_j$.

The complete set of eigenstates of the rational integrals of motion is given by the Richardson ansatz (5, 6). This fact led Gaudin¹² to try to relate his integrable models to the BCS Hamiltonian without success. The proof of integrability of the BCS Hamiltonian was found later in ref.¹³. We will not present the general solution of the two integrable families here, referring the reader to refs.⁹⁻¹¹.

The key point is that any Hamiltonian that can be expressed as a linear combination of the R operators can be treated exactly using this method. In the following sections, we discuss nuclear applications of the standard pairing model and of a new model based on the hyperbolic family.

4. Applications of the Richardson solution to pairing in nuclear physics

Richardson himself started to explore analytically the exact solution in nuclear structure for few pairs outside a doubly-magic core^{14,15}. He also proposed a numerical method to solve the equations for systems with equidistant levels¹⁶, a model that was subsequently used as a benchmark to test many-body approximations¹⁷. However, the first application of the Richardson solution to a real nuclear system was reported by Andersson and Krumlinde¹⁸ in 1977. They studied the properties of high-spin states in ^{152}Dy using an oblate deformed oscillator potential and including the effects of pairing at several different levels of approximation. They compared the results when pairing was treated with the traditional BCS approximation, when it was treated in PBCS approximation (using the saddle point approximation) and when it was treated exactly using the Richardson method.

Following that early work, there were sporadic references to the Richardson method but no realistic studies of atomic nuclei until just a few years ago. In 2007, Dussel *et al.*¹⁹ reported a systematic study of pairing correlations in the even Sm isotopes, from ^{144}Sm through ^{158}Sm , using the self-consistent deformed Hartree Fock+BCS method. The calculations made use of the density-dependent Skyrme force, SLy4, and treated pairing correlations using a pairing force with constant strength G assuming axial symmetry and taking into account 11 major shells.

Using the results at self-consistency to define the HF mean field, pairing effects within *that* mean field were then considered using the alternative number-conserving PBCS approach and the exact Richardson approach. In this way it was possible to directly compare the three approaches to pairing with the same pairing Hamiltonian, a primary focus of the study. It should be noted here that the Hilbert space dimensions associated with the residual neutron pairing Hamiltonian is of the order of 3.9×10^{53} for ^{154}Sm , whereas the exact Richardson approach requires the solution of a coupled set of 46 non-linear equations.

In the one semi-magic nucleus ^{144}Sm that was studied, the principal

correlation effects arise when projection is included, taking the system from one that is normal at the level of BCS to one with substantial pairing correlations. Treating pairing exactly provides a further modest increase in pairing correlations of about 0.3 MeV . In non-semi-magic nuclei, the effect on the pairing correlation energy of the exact solution is significantly more pronounced. While there too number projection provides a substantial lowering of the energy, it now misses about 1 MeV of the exact correlation energy that derives from the Richardson solution.

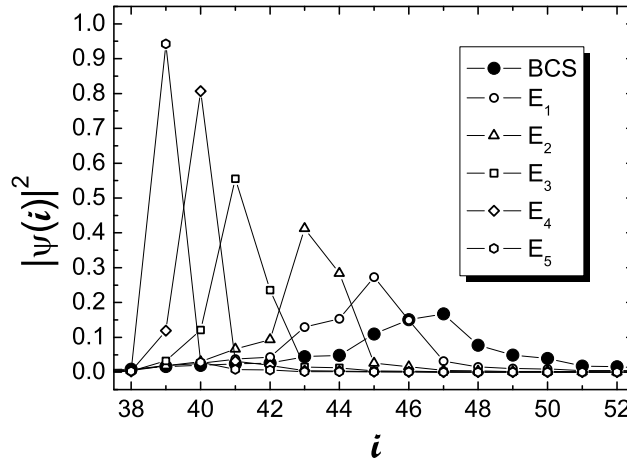


Fig. 1. Square of the wave function of the most collective Cooper pairs in ^{154}Sm (denoted E_1 , E_2 , E_3 , E_4 , and E_5) and the pairing tensor (BCS) versus the single-particle level i in the Hartree-Fock basis. The results are presented for the physical value of the pairing strength, $G = 0.106 \text{ MeV}$.

As noted earlier, the Richardson prescription gives rise to distinct Cooper pairs with distinct structure. This is illustrated in Figure 1 where we compare the square of the wave function for the most correlated Cooper pairs in ^{154}Sm , *i.e.* those whose pair energies lie farthest from the real axis in the complex plane, with the square of the pairing tensor $u_i v_i$ that derives from the corresponding BCS solution. All wave functions are plotted versus the order of the single-particle states to make clear the relevant mixing of configurations in each pair. The pair label E_1 refers to the two most collective pairs (with complex conjugate pair energies). E_2 refers to

the next two most collective pairs, which are however only marginally collective. E_3 refers to the next two in descending order of collectivity, but they only involve perturbative mixing of configurations and are not truly collective. The final two that are shown, E_4 and E_5 , have real pair energies and involve almost pure single-particle configurations. From the figure, we see that even the most collective Cooper pairs are much less collective than $u_i v_i$, and therefore that their size in coordinate space is significantly larger than that of the BCS pairing tensor²⁰, which is often used in the literature as a definition of the Cooper pair wave function²¹.

The exact Richardson solution was also used to study the gradual emergence of superconductivity in the Sn isotopes²². By making use of an exact mapping between the Richardson equations and a classical electrostatic problem in two dimensions, it was possible to get a physical picture of how superconductivity develops as a function of the pairing strength. In particular, as the pairing strength is increased the pair energies gradually merge into larger structures in the complex plane as pair correlations gradually overcome single-particle effects.

More recently, the Richardson solution has been applied to the treatment of pair correlations involving the continuum. The first work by Hasegawa and Kaneko²³ considered only the effect of resonances in the continuum and as a result obtained complex energies even for the bound states of the system. Subsequent work by Id Betan^{24,25} included the effects of the true continuum. The most recent paper²⁵ treated nuclear chains that include both bound and unbound systems, *e.g.* the even-A Carbon isotopes up to ^{28}C . When the system is bound, the pair energies that contribute to the ground state occur in complex conjugate pairs, thus preserving the real nature of the ground state energy. Once the system becomes unbound this ceases to be the case. Now the pair energies that contribute to the ground state do not occur in complex conjugate pairs, explaining how a width arises in the energy of an unbound system within the Richardson approach.

5. The hyperbolic model

The hyperbolic family of models did not find a physical realization until very recently when it was shown that they could model a p -wave pairing Hamiltonian in a 2-dimensional lattice²⁶, such that it was possible to study with the exact solution an exotic phase diagram having a non-trivial topological phase and a third-order quantum phase transition²⁷. Immediately

thereafter, it was shown that the hyperbolic family gives rise to a separable pairing Hamiltonian with 2 free parameters that can be adjusted to reproduce the properties of heavy nuclei as described by a Gogny HFB treatment²⁸. Both applications are based on a simple linear combination of hyperbolic integrals which give rise to the separable pairing Hamiltonian

$$H = \sum_i \eta_i K_i^0 - G \sum_{i,i'} \sqrt{\eta_i \eta_{i'}} K_i^+ K_{i'}^- . \quad (14)$$

If we interpret the parameters η_i as single-particle energies corresponding to a nuclear mean-field potential, the pairing interaction has the unphysical behavior of increasing in strength with energy. In order to reverse this unwanted effect, we define $\eta_i = 2(\varepsilon_i - \alpha)$, where the free parameter α plays the role of an energy cutoff and ε_i is the single-particle energy of the mean-field level i . Making use of the pair representation of $SU(2)$, the exactly-solvable pairing Hamiltonian (14) takes the form

$$H = \sum_i \varepsilon_i \left(c_i^\dagger c_i + c_i^\dagger c_i^\dagger \right) - 2G \sum_{ii'} \sqrt{(\alpha - \varepsilon_i)(\alpha - \varepsilon_{i'})} c_i^\dagger c_i^\dagger c_{i'} c_{i'} , \quad (15)$$

with eigenvalues $E = 2\alpha M + \sum_i \varepsilon_i \nu_i + \sum_\beta E_\beta$. The pair energies E_β correspond to a solution of the set of non-linear Richardson equations

$$\frac{1}{2} \sum_i \frac{1}{\eta_i - E_\beta} - \sum_{\beta' (\neq \beta)} \frac{1}{E_{\beta'} - E_\beta} = \frac{Q}{E_\beta} , \quad (16)$$

where $Q = \frac{1}{2G} - \frac{L}{2} + M - 1$. Each particular solution of Eq. (16) defines a unique eigenstate.

Due to the separable character of the hyperbolic Hamiltonian, in BCS approximation the gaps $\Delta_i = 2G\sqrt{\alpha - \varepsilon_i} \sum_{i'} \sqrt{\alpha - \varepsilon_{i'}} u_{i'} v_{i'} = \Delta \sqrt{\alpha - \varepsilon_i}$ and the pairing tensor $u_i v_i = \frac{\Delta \sqrt{\alpha - \varepsilon_i}}{2\sqrt{(\varepsilon_i - \mu)^2 + (\alpha - \varepsilon_i)\Delta^2}}$ have a very restricted form. In order to test the validity of the exactly solvable Hamiltonian (15) we take the single-particle energies ε_i from the HF energies of a Gogny HFB calculation and we fit the parameters α and G to the gaps and pairing tensor in the HF basis. Figure 2 shows the comparison for protons in ^{238}U between the Gogny HFB results in the HF basis and the BCS approximation of the hyperbolic model. From these results we extracted the values $\alpha = 25.25 \text{ MeV}$ and $G = 2 \times 10^{-3} \text{ MeV}$. The valence space determined by the cutoff α corresponds to 148 levels with 46 proton pairs. The size of the Hamiltonian in this space is 4.83×10^{38} , well beyond the limits of exact diagonalization. However, the integrability of the hyperbolic model provides an exact solution by solving a set of 46 non-linear coupled equations.

Moreover, the exact solution shows a gain in correlation of more than 2 *MeV* suggesting the importance of taking into account correlations beyond mean-field.

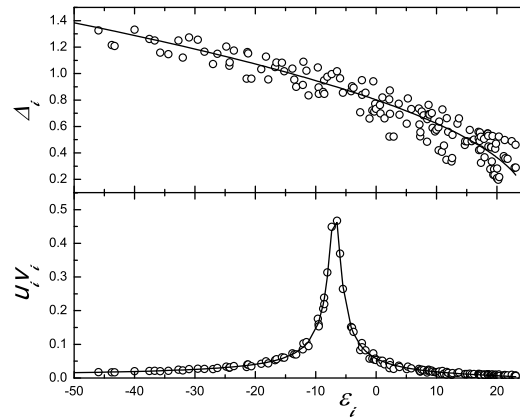


Fig. 2. Gaps Δ_i and pairing tensor $u_i v_i$ for protons in ^{238}U . Open circles are Gogny HFB results in *MeV*. Solid lines are BCS results of the hyperbolic Hamiltonian in *MeV*.

6. Extensions to non-compact and higher rank algebras

Up to now, we have restricted our discussion to RG models that are based on the compact rank-1 $SU(2)$ pair algebra. The method of constructing RG models can be extended to the non-compact rank-1 $SU(1, 1)$ algebra as well, whereby pairing in bosonic systems²⁹ is described in complete analogy with the $SU(2)$ case. An early application to the $SO(6)$ to $U(5)$ line of integrability of the Interacting Boson Model (IBM) was reported in ref.³⁰, with the exact solution being obtained there directly using an infinite dimensional algebraic technique. Further work on the IBM using the integrable $SU(1, 1)$ RG model³¹ including high-spin bosons (d, g, \dots) revealed a particular feature of the repulsive boson pairing interaction that seems to provide a new mechanism for the enhancement of $s - d$ dominance, giving further support for the validity of the $s - d$ Interacting Boson Model.

The RG models are not constrained to rank-1 algebras. They can be extended to any semi-simple Lie algebra³². Richardson himself studied some restricted solutions of the $T=1$ pairing model³³ and the $T=0, 1$ pair-

ing model³⁴. As a general statement, the reduced pairing Hamiltonian is exactly solvable for any multi-component system. The first step in finding an exact solution is to identify the Lie algebra of the commuting pair operators and then to specialize the general solution given in³². One has to keep in mind that while the $SU(2)$ RG model has a single set of unknown parameters, the pair energies, larger rank algebras have as many sets of unknown parameters as the rank of the algebra. Therefore, the higher the rank of the algebra, the greater is the complexity of the solution. Several pairing Hamiltonians with relevance to nuclear physics have been studied in the last few years.

i. The rank-2 $SO(5)$ RG model³⁵ describes $T=1$ proton-neutron pairing with non-degenerate single particle levels. The exact solution has two sets of spectral parameters, the pair energies and a second set associated with the $SU(2)$ isospin subalgebra. In spite of the greater complexity, it was possible to solve exactly a $T=1$ pairing Hamiltonian for the nucleus ^{64}Ge using a ^{40}Ca core, with a Hilbert space dimension well beyond the limits of exact diagonalization.

ii. The rank-3 $SO(6)$ RG model³⁶ describes color pairing, *i. e.* pairing between three-component fermions. The exact Richardson equations have three sets of spectral parameters, of which one correspond to the pair energies and the other two are responsible for the different couplings within the $SU(3)$ color subalgebra. The model has been used to study the phase diagram of polarized three-component fermion atomic gases. However, it could in principle be exploited to describe non-relativistic quark systems.

iii. With increasing complexity, the rank-4 $SO(8)$ RG model³⁷ describes either $T=0,1$ proton-neutron pairing or four-component fermion gases. It contains four sets of spectral parameters. The model has been used to study alpha-like structures represented by clusters in the parameter space, and how these clusters dissolve into like-particle pairs with increasing isospin.

iv. The rank-2 non-compact $SO(3,2)$ algebra generalizes the bosonic RG models to systems of interacting proton and neutron bosons³⁸. The model describes the IBM2 in the line of integrability between vibrational and γ -soft nuclei. The exact solution has been employed to study the influence of high-spin f and g bosons in the low-energy spectrum

7. Summary and future outlook

In this article, we have reviewed Richardson's solution of the pairing model and have discussed its generalization to a wider class of exactly solvable

models. We have also discussed the application of these models to a variety of problems in nuclear structure physics in which pairing plays a role. It should be noted here, however, that all of the models that we have discussed are restricted to the pairing degree of freedom and thus do not allow explicit treatment of deformation effects. It is only through the use of Nilsson or deformed Hartree Fock single-particle energies that effects of deformation are simulated.

A key feature of the Richardson-Gaudin integrable models, is that they transform the diagonalization of the hamiltonian matrix, whose dimension grows exponentially with the size of the system, to the solution of a set of M coupled non-linear equations where M is the number of pairs. This makes it possible to treat problems that could otherwise not be treated and in doing so to obtain information that is otherwise inaccessible. For example, we reported an application of the rational RG pairing model to the even-mass Sm isotopes, where the size of the Hilbert space would exceed 10^{53} states, and an application of the hyperbolic RG pairing model to U^{238} , where the size of the Hilbert space would exceed 10^{38} states. In both cases, substantial gains in correlation energy were found when the problem was treated exactly.

The exactly solvable RG Hamiltonians also provide excellent benchmarks for testing approximations beyond HFB in realistic situations both for even-even and odd-mass nuclei. Moreover, a self-consistent HF plus exact pairing approach could in principle be implemented to describe large regions of the table of nuclides. It might be possible to extend such a self-consistent approach to the O(5) RG model, providing in this way a better description of those nuclei with $N \sim Z$ in which T=1 proton-neutron pairing correlations are expected to play a significant role. Unfortunately, the SO(8) T=0,1 RG model cannot accommodate the spin-orbit splitting in the single-particle energies. Nevertheless, this model could play an important role in helping to understand quartet clusterization and quartet condensation in nuclear and cold atom systems. Finally, extension of the RG models to include the effects of the continuum seems to be an especially promising avenue to explore the physics of weakly-bound nuclei.

Acknowledgments

This work was supported in part by the Spanish Ministry for Science and Innovation under Project No. FIS2009-07277 and the National Science Foundation under Grant No. PHY-0854873.

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